Resonant tunneling in fractional quantum Hall effect: superperiods and braiding statistics

Jainendra K. Jain and Chuntai Shi

Department of Physics, 104 Davey Laboratory, The Pennsylvania State University, Pennsylvania, 16802

We study theoretically resonant tunneling of composite fermions through their quasi-bound states around a fractional quantum Hall island, and find a rich set of possible transitions of the island state as a function of the magnetic field or the backgate voltage. These considerations have possible relevance to a recent experimental study, and bring out many subtleties involved in deducing fractional braiding statistics.

Properties of an electron system in a doubly connected geometry are periodic in the flux through the region devoid of electrons, with the period being precisely one flux quantum[1] (flux quantum is defined as $\phi_0 = hc/|e|$). An example is Aharonov Bohm oscillations in the resistance of a ring as a function of the magnetic field. Periods smaller than ϕ_0 are in principle possible, and do occur in superconducting rings. It was noted in Ref. 2 that the fractional quantum Hall effect[3] (FQHE) allows for the possibility of superperiods $K\phi_0$ with $K > 1$. The geometry considered therein contains an island of ν_1 FQHE state on the background of a ν_0 FQHE state, and the periodic behavior occurs with respect to the magnetic flux through the island. Given the singly connected geometry, there is no reason, *a priori*, why there ought to be any period, ϕ_0 or otherwise. However, as a result of the incompressibility of the two Hall states, the minimal readjustment of the island is accompanied by a change of an integral number of flux quanta through it, producing, possibly, a superperiod. For example, for a $2/5$ island surrounded by the $1/3$ sea, a period of $5\phi_0$ was predicted[2].

The unusual period is closely related to the fact that the charge of "quasiparticles" of the FQHE state is fractionally quantized[4]. It is believed that FQHE quasiparticles also obey fractional braiding statistics[5,6], which refers to the property that the Berry phase associated with a closed loop of a quasiparticle changes, upon insertion of another quasiparticle inside the loop, by $2\pi\theta^*$, with $\theta^* \neq$ integer[7,8]. Fractional braiding statistics was proposed as a theoretical construct in late 1970's[5,6], and FQHE quasiparticles are presently the only viable candidates for its realization. While interesting in its own right, a definitive experimental observation of "abelian" braiding statistics would also appear to be a necessary first step on the way to the more complicated "non-abelian" braiding statistics believed to occur in paired composite fermion states, which has attracted much attention recently[9].

A very interesting recent experiment of Camino, Zhou, and Goldman[10] has reported quasiperiodic peaks in the FQHE regime. They argue that the peaks occur due to resonant tunneling through quasi-bound states[11] around a $2/5$ island surrounded by the $1/3$ sea, and estimate that the period corresponds to a flux change of $5\phi_0$ through the island as a function of the magnetic field, and to a change of charge two (in units of the electron charge $e = -|e|$) in the island as a function of the backgate potential $V_{\rm BG}$. They interpret this result as providing a direct observation of fractional braiding statistics. The reasoning, briefly, is as follows[10]. The Berry phase acquired by a "test" quasiparticle for a path encircling the island of area A is assumed to change by 2π between two successive resonant tunnelings. Modeling the test quasiparticle as an object with charge e^* and braiding statistics θ^*, this gives[12], with $\phi_0^* = hc/|e^*| = (|e/e^*|)\phi_0$,

$$-2\pi\frac{\Delta(BA)}{\phi_0^*} + 2\pi\theta^*\Delta N_q = \pm 2\pi \ . \tag{1}$$

Here $\Delta(BA)$ is change in the flux and ΔN_q is the change in the number of quasiparticles enclosed by the path between two successive tunneling resonances. Substituting $\Delta(BA) = 5\phi_0$ and $e^*/e = 1/3$ (as appropriate for a negatively charged test quasiparticle in the $1/3$ region) gives $\theta^* = 2/(3\Delta N_q)$ or $\theta^* = 8/(3\Delta N_q)$. Identifying charge two with ten charge $1/5$ quasiparticles in the $2/5$ island ($\Delta N_q = 10$) yields $\theta^* = 1/15$ or $4/15$, which was interpreted as the *relative* braiding statistics of the $e^*/e = 1/3$ quasiparticle in the $1/3$ state with respect to an $e^*/e = 1/5$ quasiparticle in the $2/5$ island.

Our theoretical analysis of the geometry of Ref. 10 shows that, because of a rich set of possible transitions of the island as well as of the island edge effects, an interpretation of the experimental results can be quite subtle and complicated. An alternative theoretical interpretation leads to an unacceptable value for the braiding statistics, and there does not seem to be any fundamental reason why the flux and charge cannot change by the smallest possible units. We speculate, at the end, on one possible scenario for reconciling theory and experiment, but much further work will be needed for a definitive understanding.

Transitions of a FQHE island: The evolution of a FQHE island is rather complicated, but various possibilities can be enumerated in the composite fermion (CF) theory[13]. Composite fermions are bound states of electrons and an even number (taken to be two below) of quantized vortices, and experience a reduced effective magnetic field. They form Landau-like levels in this reduced magnetic field, which will be called "Λ levels."[14] We will consider an island surrounded by the FQHE sea

TABLE I: The change in flux, $\frac{\Delta(BA)}{\phi_0}$, the total charge, ΔQ, and the excess charge, Δq, associated with the transition of the FQHE island $[N_1]$ (notation explained in text) into various final states.

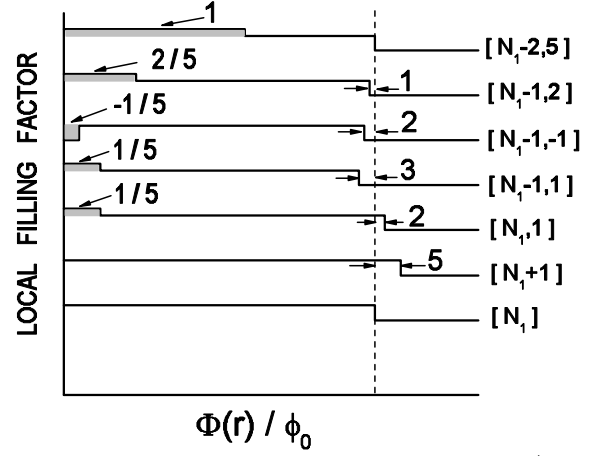| final state	$\frac{\Delta(BA)}{\phi_0}$	ΔQ	Δq
$[N_1+1]$	$2n+3$	$n+1$	$1/(2n+1)$
$[N_1,1]$	2	1	$1/(2n+1)$
$[N_1-1]$	$-2n-3$	$-n-1$	$-1/(2n+1)$
$[N_1,-1]$	-2	-1	$-1/(2n+1)$
$[N_1-1,1]$	$-2n-1$	$-n$	0
$[N_1-1,n+1]$	-1	0	$n/(2n+1)$
$[N_1-2,2n+3]$	0	1	1

FIG. 1: Schematic view of several transitions of the 2/5 island discussed in the text. The y-axis labels the local filling factor as a function of the distance r from the center, with different traces offset for clarity (the filling factor on the far right is 1/3 for each trace). The x-axis is the flux through the area enclosed by the disk of radius r ($\Phi(r) = \pi r^2 B$) in units of the flux quantum ϕ_0. The bottom plot labeled $[N_1]$ shows a 2/5 island inside the 1/3 state, with N_1 composite fermions in the second Λ level. Addition of a composite fermion at the boundary in the second Λ level (at the center in the third Λ level) produces $[N_1+1]$ ($[N_1,1]$), and moving a composite fermion from the boundary to the center results in $[N_1-1,1]$. The top two trace show the density for $[N_1-2,5]$ and $[N_1-1,2]$. The numbers near the shaded regions show the excess charge there. The numbers near the vertical dashed line show the shift in the island edge in units of flux quanta. The density oscillations at the edge and near the charge 1/5 CF-quasiparticle have been suppressed for simplicity. For ease of illustration, the island has been taken to be circularly symmetric, and all quasiparticles are added at the center; changes in area and charge given in the text are more generally valid.

at $\nu_0 = n/(2n+1)$, which has $\nu_0^* = n$ filled Λ levels. The island state will be denoted by $[N_1, N_2, \cdots]$, where N_j is the number of composite fermions in the $(n+j)^{\text{th}}$ Λ level. In particular, $[N_1]$ represents an island of the $\nu_1 = (n+1)/(2n+3)$ (i.e. $\nu_1^* = n+1$) FQHE state. The lowest n Λ levels will be taken to be fully occupied and inert in the entire region of interest, and the number of composite fermions in them (N_0) will be suppressed for notational convenience. We shall assume here the simplest model, neglecting, in particular, the effect of screening at the edge of the island; while one can see such complications destroying simple quasi-periodic behavior, it is difficult to imagine how they could be responsible for it.

A "quasiparticle" of a FQHE state is an *isolated* composite fermion in an excited Λ level, and a "quasihole" is a missing composite fermion from an otherwise full Λ level[13]. The localized charge *excess* (i.e., the charge sticking out of the uniform incompressible state; also called the "local" charge[15]) associated with a CF-quasiparticle of the $\nu_0 = n/(2n+1)$ FQHE state is $1/(2n+1)$ in units of the electron charge[13]. When CF-quasiparticles are far apart, they have well defined fractional braiding statistics with $\theta^* = 2/(2n+1)$ (Refs. 13,15). When they are overlapping, however, the braiding statistics is not a meaningful concept; that is intuitively obvious, and also confirmed by explicit calculation[16,17,18]. The braiding statistics of CF-quasiparticles is not an additional concept, but a consequence of the physics encoded in the CF theory, and follows from the topological aspect that composite fermions carry an even number of quantized vortices[16]. The CF theory reproduces the earlier result[8] for quasiholes at $\nu = 1/3$, and enables a microscopic evaluation of θ^* for quasiparticles of 1/3 and other FQHE states[16,18].

What gives a discrete character to the island state is that it can change only through integral variations in the number of composite fermions occupying various Λ levels. In deducing the corresponding changes in the electronic state, we make use of the following results: In the ν_1 state: (i) the *total* charge per flux quantum is $\nu_1 = (n+1)/(2n+3)$, whereas (ii) the *excess* charge per flux quantum is $\nu_1 - \nu_0 = [(2n+1)(2n+3)]^{-1}$. (iii) From

the perspective of the "substrate" FQHE state (ν_0), a composite fermion in any higher Λ level has an *excess* charge of $1/(2n+1)$.

The two "elementary" transitions of a FQHE island are given by the addition of a composite fermion to the island edge ($[N_1] \rightarrow [N_1+1]$) or in a higher Λ level in the island interior ($[N_1] \rightarrow [N_1,1]$). The excess charge in either case is $\Delta q = 1/(2n+1)$, because a single CF-quasiparticle has been added. The change in the total charge in the island, ΔQ, and the flux passing through it, $\Delta(BA)/\phi_0$, are determined as follows. In the former case the island accommodates the excess charge by expanding to enclose $2n+3$ additional flux quanta, and the total charge increases by $\Delta Q = (2n+3)\nu_1 = n+1$. In the latter case, the excess charge appears in two places: $1/(2n+3)$ in the interior of the ν_1 island, equal to the local charge of the quasiparticle for this state, and $2/[(2n+1)(2n+3)]$ at the boundary. [By definition, when a composite fermion is added in the interior of the island, it shifts the island edge by an amount that encloses two additional flux quanta, giving an excess boundary charge of $2(\nu_1-\nu_0)$.] The total island charge changes by $\Delta Q = 1$ for the latter case, with

$2\nu_1$ coming from the island edge and $1/(2n+3)$ from the interior quasiparticle. The $[N_1] \rightarrow [N_1, 1]$ transition illustrates how apparently simple transitions of composite fermions manifest through rather complicated, nonlocal changes in the electronic state.

$\Delta(BA)$, ΔQ and Δq can be similarly determined due to the removal of a composite fermion from the island edge or interior, or resulting from a combination of several elementary transitions. Many transitions are listed in Table 1 and depicted schematically in Fig. 1 for the special case of a 2/5 island on the 1/3 substrate. We have also confirmed the basic physics presented in Fig. 1 by extensive calculations with explicit microscopic wave functions for composite fermions.

It may be checked that the transitions in Table 1 are, in general, described by the equation[12]

$$-2\pi\frac{\Delta(BA)}{\phi_0^*} + 2\pi\theta^*\Delta N_q = 2k\pi \; , \qquad (2)$$

with $\theta^* = 2/(2n+1)$, the braiding statistics of the CF quasiparticles of the ν_0 state[16], and $\Delta N_q = r - s$ for the transition $[N_1] \rightarrow [N_1 - s, r]$. The values of k are given by $-1, 0, 1, 0, 1, 1, 2$, for the transitions in Table 1, from top to bottom, respectively. The Berry phase change between successive resonant tunnelings can be zero (with the changes in the Aharonov-Bohm and the statistical phases canceling one another), and, under certain constraint, even 4π (an example is given below).

The actual island is likely to be more complicated than that considered above. Depending on the shape of the potential due to confinement *and* disorder, many Λ levels may be occupied and some localized quasiparticles and quasiholes may be present. For example, 3/7 hills or 1/3 lakes can exist inside the 2/5 island. The presence of such non-idealities does not affect the conclusions regarding changes in the size and the charge of the island due to addition or removal of a composite fermion at the edge or in the interior. Also, transfer of a composite fermion from one localized state to another within the interior of the island does not change either its area or its charge.

Relevance to experiment: Returning to the experiment of Ref. 10, the creation of a charge 1/5 particle in the interior of the 2/5 island has associated with it a charge 2/15 at the island edge, which must also be accounted for in the Berry phase calculation. It is therefore more appropriate to view the charge two as six charge 1/5 quasiparticles in the island interior plus six units of charge 2/15 at the island edge. From the vantage point of the test quasiparticle, this is equivalent to six charge 1/3 quasiparticles. Substituting $e^*/e = 1/3$, $\Delta(BA) = 5\phi_0$, and $\Delta N_q = 6$ into Eq. (1) gives $\theta^* = 1/9$ or $4/9$, which now is the braiding statistics for the 1/3 quasiparticles relative to one another. This value in disagreement with the theoretically accepted one. Further insight is gained by noting that, for exterior Berry trajectories, a collection of six charge 1/3 quasiparticles (or, for that matter, ten charge 1/5 quasiparticles) is topologically indistinguishable from two charge one electrons. This points, on

the one hand, to a conceptual difficulty with interpreting the result as a measure of the relative braiding statistics of *fractionally charged* quasiparticles; on the other, $\Delta N_q = 2$ produces $\theta^* = 4/3$ or $1/3$ for the braiding statistics of the 1/3 quasiparticle relative to an electron, which also contradicts theoretical result[19].

From a microscopic point of view, the experimental result seems, at first, to correspond to the elementary transition $[N_1] \rightarrow [N_1 + 1]$ (considered in Ref. 2) for which the island area increases by $5\phi_0$ and its net charge by two units ($\Delta Q = 2$). (Note that $\Delta N_q = 1$ for this transition, because a single composite fermion has been added to the boundary of the second Λ level.) The value $\Delta Q = 2$ appears to be consistent with the observed period in backgate voltage. However, ΔV_{BG} is proportional to the change in the *excess* charge, $\Delta q = 1/3$ (table 1), and not to ΔQ, part of which is due to the increase in the area of the island and counts the charge in the 1/3 substrate that was already present. A change of $\Delta q = 1/3$ between two successive ΔV_{BG} is presumably too small for the experimental parameters of Ref. 10.

There, however, is no reason why the island transitions as a function of V_{BG} should be the same as those as a function of B, as assumed implicitly above. [Should the transitions be different, only one of ΔN_q and $\Delta(AB)$ in Eq. 1 is known for each transition, preventing a determination of θ^*.] The two experiments ought to be analyzed independently. It is also necessary to allow for the possibility of more complex combinations of elementary transitions, which can occur due to energetic considerations arising from the electrostatics of the problem. It seems reasonable that, as a function of B, the charge on the 2/5 island does not change during consecutive transitions (although it is redistributed internally), because that would have a large Coulomb blockade energy associated with it[10]. With what minimum flux change can the island readjust while preserving the total charge inside it? Clearly, that must involve addition of composite fermions in the interior of the island accompanied by removal of composite fermions from its edge. The smallest flux change is ϕ_0, and the associated transition is into $[N_1 - 1, n + 1]$ (table I). Consider next variation of V_{BG}. What is the smallest unit of charge that can be added to the island (at a fixed B) without altering its area? One can convince oneself that the transition is into $[N_1 - 2, 2n + 3]$, which implies $\Delta Q = \Delta q = 1$. Both these results are obvious in the compact spherical geometry. The flux through a uniform density ν_1 FQHE state on a sphere (representing the island only) can be changed by ϕ_0 without altering the area or the number of particles, at the cost of creating quasiparticles. Further, a unit charge can surely be added to any FQHE state without changing either the size or the magnetic field (the "electron" goes into a complicated excited state of composite fermions[20]). In light of this general argument, it is not obvious theoretically why the flux through the island should change in units of $5\phi_0$ and charge in units of two. Incidentally, a transition $[N_1] \rightarrow [N_1 - 2, 2n + 3]$

corresponds to a change of Berry phase by 4π ($k = 2$ in Eq. 2); this "superselection rule" arises from the constraint of fixed area.

We do not know, at this stage, how to reconcile theory and experiment. Several parameters (e.g. the existence or the area of the 2/5 island; the filling factor variation within the island; the relation between V_{BG} and the charge on the island) are deduced indirectly in experiment, making an unambiguous conversion of ΔB and ΔV_{BG} periods into flux and charge periods difficult. One can therefore ask if the experiment may be consistent with other transitions mentioned above. It is appealing to consider the possibility of the smallest periods. A ϕ_0 period as a function of B would imply, for the experiment of Ref. 10, an island area of $A = 0.2 \times 10^{-8}$ cm^2, which is much smaller than either the lithographic area of the island (3.5×10^{-8} cm^2) or the area corresponding to the ΔB period at $\nu = 1$ (1.5×10^{-8} cm^2). However, a smaller area could perhaps occur either if the peak density in the island is less than the density in the unpatterned sample, or if many disconnected patches of 2/5 exist (as a result of disorder) rather than a singly connected island, with one of them dominating the resonant tunneling process. The smallest period in V_{BG} is produced by the transition $[N_1] \rightarrow [N_1 + 1]$ or $[N_1] \rightarrow [N_1, 1]$, which slightly alters the island area but adds the smallest unit of excess charge ($\Delta q = 1/3$). That corresponds, for the area $A = 0.2 \times 10^{-8}$ cm^2, to $d\rho/dV_{\text{BG}} = 1.7 \times 10^8$ cm^{-2}/V, which compares favorably to the values at $B = 0$ and $\nu = 1$ (2.4×10^8 cm^{-2}/V and 2.0×10^8 cm^{-2}/V, respectively). Obviously, further work will be needed to clarify the situation. It is noted that possible non-equilibrium or long-time scale effects have been neglected in our analysis.

We close with general comments regarding the relation of such a tunneling experiment to braiding statistics, independent of which transition is responsible for the quasiperiodicity. While all transitions can be consistently interpreted in terms of particles with a braiding statistics of $\theta^* = 2/(2n+1)$ (which is the braiding statistics of CF-quasiparticles of the substrate FQHE state), as evident from the discussion near Eq. 2, one may question, for the following reasons, if any of them can be taken as providing a definitive observation of braiding statistics. For an unambiguous measurement of the braiding statistics it is crucial that the test quasiparticle be well separated from all of the quasiparticles it is encircling. For the transition $[N_1] \rightarrow [N_1 + 1]$, the composite fermion added at the edge does not qualify as a quasiparticle, but is a part of the ν_1 state (see, for example, the $[N_1 + 1]$ trace in Fig. 1). For transitions into $[N_1, 1]$ or $[N_1 - 1, n + 1]$, the quasiparticles have an induced part residing at the edge, which interferes with the test composite fermion at the island edge, making the braiding statistics ill-defined[17,18,21]. How about the transition $[N_1] \rightarrow [N_1 - 2, 2n+3]$, where the induced charge at the island boundary has been explicitly removed, leaving only $2n + 3$ charge $1/(2n + 3)$ quasiparticles in the interior? This also does not enable a measurement of the braiding statistics of quasiparticles because, together, the interior quasiparticles behave as a single electron in their topological properties. It may also be noted that the period can be derived in all cases solely from the knowledge of the values of fractional charges and filling factors in the island and the exterior regions.

[1] N. Byers and C.N. Yang, Phys. Rev. Lett. **7**, 46 (1961).

[2] J.K. Jain, S.A. Kivelson, and D.J. Thouless, Phys. Rev. Lett. **71**, 3003-3006 (1993).

[3] D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[4] R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[5] J.M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977).

[6] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).

[7] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[8] D. Arovas, J.R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[9] See, for example, Physics Today **58** (10), 21 (2005).

[10] F.E. Camino, W. Zhou, and V.J. Goldman, Phys. Rev. B **72**, 075342 (2005).

[11] J.K. Jain and S.A. Kivelson, Phys. Rev. Lett. **62**, 231 (1989).

[12] S. Kivelson, Phys. Rev. Lett. **65**, 3369 (1990).

[13] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); J.K. Jain, Phys. Rev. B **41**, 7653 (1990), Phys. Rev. B **40**, R8079 (1989).

[14] These are also called quasi- or CF-Landau levels.

[15] A.S. Goldhaber and J.K. Jain, Phys. Lett. A **199**, 267 (1995).

[16] G.S. Jeon, K.L. Graham, J.K. Jain, Phys. Rev. Lett. **91**, 036801 (2003); Phys. Rev. B **70**, 125316 (2004).

[17] H. Kjønsberg and J. Myrheim, Int. J. Mod. Phys. A **14**, 537 (1999).

[18] H. Kjønsberg and J.M. Leinaas, Nucl. Phys. B **559**, 705 (1999).

[19] The relative statistics of a charge 1/3 quasiparticle with respect to an electron is $\theta^* = 2$, following the arguments of Ref. 16, or from Eq. 2 for the bottom transition in Table 1. Equating the electron to five charge 1/5 quasiparticles allows a determination of the relative braiding statistics of a charge 1/3 quasiparticle in the 1/3 state with respect to a charge 1/5 quasiparticle in the 2/5 state to be $\theta^* = 2/5$.

[20] J.K. Jain and M.R. Peterson, Phys. Rev. Lett. **94**, 186808 (2005).

[21] A.S. Goldhaber, J.K. Jain, and G.S. Jeon, Phys. Rev. Lett. **93**, 169705 (2004).